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Brief paper

Distributed finite-time estimation of the bounds on algebraic connectivity for directed graphs☆

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ABSTRACT

This paper studied distributed estimation of the bounds on algebraic connectivity for a directed graph (i.e., digraph). As is well known, the main challenge of the underlying problem is how to enable local awareness of an entity otherwise prone to global information, in the presence of communication topology. More specifically, we introduce a novel state-dependent approach to estimate the bounds on algebraic connectivity with mild requirement on topology and communication effort. Compared with existing results, the proposed algorithm does not estimate eigenvalues or eigenvectors directly, rather it exploits their implications on the consensus procedure, and achieves a tradeoff between estimation accuracy and topological/communication requirement, and its convergence can be expected in a finite time. Simulation results verified the performance of the proposed strategy.

1. Introduction

In the past decade, distributed estimation of network connectivity became an attractive topic of research in the community (Garin & Schenato, 2010), and its challenge is how to estimate one global criterion (i.e., eigenvalue, eigenvector) at each system using only the local measurements. In particular, the most commonly estimated criteria include the second smallest eigenvalue of graph Laplacian (commonly referred to as algebraic connectivity or Fiedler value for an undirected graph (Fiedler, 1973)) and its associated right eigenvector (also known as Fiedler vector), or the first left eigenvector (i.e., FiLE (Punzo, Young, Macdonald, & Leonard, 2016; Qu, Li, & Lewis, 2014; Yu & Li, 2017)). More specifically, Fiedler vector often serves as an index to partition an undirected graph (Chung, 1997), and the FiLE is proved to be instrumental in describing connectivity of a digraph (Punzo et al., 2016; Qu et al., 2014), while algebraic connectivity preserves the most prominent properties (i.e., the synchronicity and convergence rate) of the entire network (Qu, 2009; Wu, 2005). Hence, it would be beneficial for the community if algebraic connectivity can be known locally in the absence of global information, preferably at mild expense of computation and communication complexity.

As is well established, power method and its closest variants are the most adopted protocol in solving the underlying problem. In particular, the arguably most effective schemes for undirected graph are done in Aragues, Shi, Dimarogonas, Sagues, and Johansson (2012), Aragues et al. (2014), Li and Qu (2013), Kempe and McSherry (2008), Sabattini, Secchi, and Chopra (2015) and Yang et al. (2010). More specifically, Yang et al. (2010) introduced a continuous time power approach to estimate the components of the Fiedler eigenvector and subsequently the eigenvalue of an undirected graph; Aragues et al. (2012) and Aragues et al. (2014) proposed a distributed scheme to estimate power of the adjacency matrix associated with an undirected graph, so as to calculate the upper and lower limits of the algebraic connectivity; Li and Qu (2013) extended the work in Yang et al. (2010) to estimate the algebraic connectivity of a strongly connected digraph with a discrete power strategy, and its enhancement can be found in Kempe and McSherry (2008); Poonawala and Spong (2015) initiated a decentralized iteration approach to compute the leading k eigenvectors of graph Laplacian using an orthogonal iteration technique; Sabattini et al. (2015) proposed a iteration-based estimation scheme for strongly connected balanced digraph. In addition, Fast Fourier Transfer (i.e., FFT) is another widely acclaimed approach in this venue. For instance, Franceschelli, Gasparri, Giua, and Seatzu (2009) and

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Franceschelli, Gasparri, Giua, and Seatzu (2013) converted the underlying problem to signal processing problem using FFT by constructing oscillators whose states oscillate at frequencies corresponding to the eigenvalues to be estimated; Other notable breakthroughs include (Gusrialdi & Qu, 2017; Tran & Kibangou, 2015; Zareh, Sabattini, & Secchi, 2018), where Tran and Kibangou (2015) proposed a novel strategy to estimate eigenvalues of Laplacian matrix by solving the factorization of the averaging matrix; Zareh et al. (2018) introduced a distributed gradient based approach to estimate all spectrum and eigenvectors of graph Laplacian; Gusrialdi and Qu (2017) enabled local awareness of graph Laplacian using an inverse based approach such that all eigenvalues can be calculated distributively.

However, it should be pointed out that all of the aforementioned results, albeit their effectiveness, are often subject to stringent topological requirements and/or excessive network communication complexity[1] (i.e., NCC). For instance, Aragues et al. (2012), Franceschelli et al. (2009, 2013), Kempe and McSherry (2008), Sabattini et al. (2015), Tran and Kibangou (2015) and Yang et al. (2010) assume the graph be undirected/balanced and connected, and Gusrialdi and Qu (2017), Li and Qu (2013) dictate the digraph should be strongly connected and the proposed algorithms work with a NCC of at least n^2 (i.e., n being the total number of agents). Moreover, convergence of existing algorithms is mostly asymptotic, implying that it will be difficult to execute them in parallel with other protocols (i.e., event-based consensus, prescribed-time network control, etc.) that are susceptible of real time information on algebraic connectivity, whose bounds (i.e., lower or upper limit) are nevertheless sufficient in some applications. Hence, it is not only theoretically interesting but also practically useful to study how to estimate, preferably within a finite time, the bounds on algebraic connectivity of a digraph, which, to the best of our knowledge, has not received sufficient attention in the community. In this paper, we attempt to tackle this problem from a fresh perspective and answer the following two simple questions: (i) is there a simple distributed scheme that provides an acceptable estimation of the said bounds for any digraph with the mild NCC and topological requirement? (ii) is there a quantitative relation between estimation accuracy and convergence rate? In what follows, we will demonstrate that these two questions can be addressed by exploiting the implications of the algebraic connectivity on the consensus process, instead of estimating or challenging the eigen-structure directly.

2. Problem formulation

In this paper, we consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ denotes node set and $\mathcal{E}$ is the set of directed edges. In particular, neighborhood set for node i is defined as $\mathcal{N}_i \subseteq \mathcal{V}$, and $(j, i) \in \mathcal{E}$ for any $j \in \mathcal{N}_i$. The adjacency matrix $A(\mathcal{G})$ is weighted and normalized[2] as Li and Qu (2014) and Peng, Li, and Ye (2018):

$$[A(\mathcal{G})]_{ik} = \begin{cases} a_{ik} > 0 & \text{if } k \in \mathcal{N}_i \\ 1 - \sum\limits_{k \in \mathcal{N}_i} a_{ik} & \text{if } k = i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $0 < a_{ij} \le 1$ for any $j \in \mathcal{N}_i$.

It follows that $A(\mathcal{G})$ is chosen to be nonnegative and row-stochastic, and its corresponding weighted graph Laplacian is

$$L = \mathbf{I} - A(\mathcal{G}) \tag{2}$$

where $\mathbf{I}$ is the identity matrix of proper dimension.

[1] In this paper, network communication complexity is defined by the number of scalars each agent has to exchange with its connected neighbors at one iteration.

[2] Note that the proposed strategy works only for normalized adjacency matrix.

By definition, it is clear that the smallest eigenvalue of Laplacian matrix L is zero, i.e., $\lambda_1(L) = 0$, corresponding to a right eigenvector $\nu_1 = \mathbf{1}_n/\sqrt{n}$ with $\mathbf{1}_n$ as a vector of ones and left eigenvector γ_1, which is also known as the first left eigenvector defined by

$$\gamma_1^T L(\mathcal{G}) = 0, \quad \gamma_1^T \mathbf{1}_n = 1 \tag{3}$$

where superscript T denotes the matrix transpose.

In what follows, eigenvalues of L are defined and sorted as $0 = \lambda_1 < |\lambda_2| \le \cdots \le |\lambda_n|$, with $|.|$ denoting the absolute value operation. As is well established, $\lambda_2(L)$ preserves the critical connectivity information of $\mathcal{G}$. More specifically, λ_2 is often known as Fiedler value and commonly referred to as the algebraic connectivity if the underlying graph is undirected. However, there is no consensus on definition of algebraic connectivity for a digraph, since it admits not only the eigenvalues, but also the state and which eigen-space it belongs to (Qu et al., 2014), and its distributed estimation is extremely challenging and computational exhaustive. Nevertheless, $\lambda_2(L)$ captures the most conservative estimate on the convergence rate of the entire network, and its distributed awareness is imperative in order to perform higher-level control and enable autonomous operations among networked systems.

In this paper, we adopt the following definition of algebraic connectivity for digraph $\mathcal{G}$ (Wu, 2005),

$$\alpha(\mathcal{G}) = |\lambda_2(L)| \tag{4}$$

It follows that definition (4) retains Fiedler's definition of algebraic connectivity, and can be applied to derive criteria for convergence rate in networks systems. In addition, (4) is reduced to conventional Fiedler value should λ_2 be real, in which case, power method or its closest variant is sufficient to enable local awareness of α, as proved vigorously in Aragues et al. (2012), Li and Qu (2013) and Yang et al. (2010). However, should $\lambda_2(L)$ be complex, definition (4) reflects the asymmetrical nature of directed information flow and incorporates the conjugate part of complex eigen-structure. Hence, the problem to be solved in this paper can be summarized as:

Problem 1. Find a distributed approach to estimate the bounds of α for digraph $\mathcal{G}$ with mild topological requirement and NCC, preferably in a finite time.

Note that, by solving Problem 1, this paper will achieve a balance among estimation error, topological requirement witnessed by the connectedness of $\mathcal{G}$, communication complexity governed by the NCC, and finite time convergence. Although plenty work has been done in the open literature, rare has addressed the said balance, which is the focus of this paper. In particular, the proposed scheme will exploit the possibility of performing distributed estimation under a generic connected digraph $\mathcal{G}$, which, to the best of our knowledge, is the least stringent topological requirement in the field of distributed connectivity estimation under a digraph. The following lemma summarizes the reducible properties of the adjacency matrix associated with a connected digraph, and its proof can be taken directly from *Lemma 4.2* of Qu (2009).

Lemma 1. *Consider graph adjacency matrix defined in (1) and under the assumption that $\mathcal{G}$ is connected or $\mathcal{G}$ has one globally reachable node. Then, matrix $A(\mathcal{G})$ is reducible, and there exists a permutation matrix Γ such that*

$$\Gamma A \Gamma^T = \left[\begin{array}{c|c} E_{11} & 0 \\ \hline E_{21} & E_{22} \end{array}\right] \tag{5}$$

where E_{11} is an irreducible block matrix. □

Remark 1. It should be pointed out that the systems corresponding to block E_{11} form a leader group.[3] If $E_{21} = 0$ and E_{22} is irreducible, the systems corresponding to blocks E_{11} and E_{22} constitute two strongly connected and yet completely isolated subgraphs, each of which can be studied separately and analogously. If $E_{21} \neq 0$ and if E_{22} is irreducible, the systems corresponding to block E_{22} form a follower group. In the case where $E_{21} = 0$ and E_{22} is reducible or in the event that $E_{21} \neq 0$ and E_{22} is reducible, the same argument can be applied to block E_{22} recursively. Hence, without loss of any generality, we assume $E_{21} \neq 0$ and the systems corresponding to block E_{11} and E_{22} form, respectively, the leader and follower group, all other scenarios can be studied analogously (Qu et al., 2014). ■

The following lemma recapitulates the instrumental role of γ_1 in describing connectivity of a digraph, its proof is a mere combination of the results from Li and Qu (2013) and Qu et al. (2014) and thus omitted here fore the sake of brevity.

Lemma 2 (*Lemma 1 and Lemma 2 of Qu et al., 2014, Lemma 2 of Li & Qu, 2013*)**.** *Consider graph Laplacian defined in (2) and its first left eigenvector γ_1 defined in (3). Then, the following statements are true:*

- *if $\mathcal{G}$ is connected, γ_1 is unique and nonnegative, its jth entry $\gamma_{1,j} = 1$ implies system j belongs to the block E_{11}, and $\gamma_{1,j} = 0$ if system j belongs to the block E_{22}, as stated in Lemma 1;*
- *if $\mathcal{G}$ is strongly connected, $\gamma_1 > 0$ is unique, and $\gamma_{1,j} > 0$ for all j;*
- *if $\mathcal{G}$ is at least connected,[4] for any $k > 0$ and $t > 0$,*

$$e^{-kLt} = \mathbf{1}_n \gamma_1^T + \Gamma_s e^{-k\Lambda_s t} W_s^T \tag{6}$$

where Λ_s is the Jordan form associated with eigenvalues λ_2 up to λ_n, $\Gamma_s \in \Re^{n\times(n-1)}$ is the normalized right eigenvector matrix by eliminating ν_1 associated with $\lambda_1(L) = 0$, and $W_s \in \Re^{n\times(n-1)}$ is its left eigenvector matrix counterpart. □

3. Main results

In this section, we attempt to solve Problem 1 using a novel state-dependent approach. The motivation is to determine the analytical connection between algebraic connectivity and instantaneous state discrepancies at each of the networked systems without exploiting topological structure directly, thus being recused from excessive communication burden and stringent topological requirement. As will be shown later, the estimation error is acceptable and upper bounded. Before moving to the main theorem, the following assumptions are made regarding the connectedness of $\mathcal{G}$ and system initialization.

Assumption 1. $\mathcal{G}$ is directed and at least connected.

Assumption 2. Each system possesses knowledge of n and its own index.[5]

In what follows, a simple distributed observer is established at system i

$$\dot{x}_i = k\sum_{j=1}^{n} a_{ij}(x_j - x_i), \quad x_i(t_0) = i \tag{7}$$

where $x_i \in \Re$ is the observer state at system i, $x_i(t_0)$ is the initial condition, a_{ij} is the (i, j)th entry of the adjacency matrix defined in (1), and $k > 0$ is the gain to be specified. Obviously, the proposed observer (7) operates at a NCC of 1. Consequently, the overall closed-loop system becomes

$$\dot{x}(t) = -kLx(t) \tag{8}$$

where $x = [x_1 \dots x_n]^T \in \Re^n$ is the overall observer state. As is well known, system (8) achieves state consensus asymptotically should Assumption 1 hold, and its final consensus value is a weighted combination of the initial conditions, namely

$$\lim_{t\to\infty} x(t) = \mathbf{1}_n \gamma_1^T x(t_0), \text{ or } \lim_{t\to\infty} x_i(t) = \gamma_1^T x(t_0), \forall i \tag{9}$$

Without loss of any generality, we assume a consensus has already been reached at time t_f if $\|x(t) - \mathbf{1}_n\gamma_1^T x(t_0)\| \leq \epsilon$ for any $t \geq t_f$ with $\epsilon > 0$ as a predefined and sufficiently small constant,[6] and $\|.\|$ denotes standard Euclidean norm. In essence, the use of observer (7) implies that the proposed estimation scheme neither challenges the eigen-structure of $L(\mathcal{G})$ directly, nor estimates an entity/state outright, as with the conventional power approaches (Garin & Schenato, 2010; Li & Qu, 2013; Yang, Freeman, & Lynch, 2008). Rather, x_i will serve as the basis to exploit the implications of α to the consensus procedure, which, under proper algebraic manipulations, leads to the recovery of α. More specifically, we attempt to reveal the quantitative relations between α and the instantaneous state discrepancies. In this regard, we define the absolute state discrepancy $\delta(t)$ as follows

$$\delta(t) \triangleq \sum_{i=1}^{n} \beta_i \left[x_i(t) - \gamma_1^T x(t_0)\right]^2 \tag{10}$$

with

$$\beta_i = \begin{cases} \gamma_{1,i} & \text{if } \mathcal{G} \text{ strongly connected} \\ 1 & \text{otherwise} \end{cases}$$

where $\gamma_{1,i}$ is the ith entry of γ_1 as defined in (3).

It follows from *Lemma 2* and *Theorem 5.20* of Qu (2009) that $\delta(t)$ is equivalent to cooperative state discrepancy and thus can be treated as a cooperative control Lyapunov function, should $\mathcal{G}$ be strongly connected. However, if $\mathcal{G}$ is connected with only one globally reachable node, communication among the leader block (i.e., E_{11}) and the follower block (i.e., E_{22}) is unidirectional, the consensus seeking procedure reflects the interactions among the leaders and the followers, while the disparity among followers is completely disregarded since they do not contribute to the procedure or $\delta(t)$. In what follows, we demonstrate that $\delta(t)$ is instrumental in studying networked control systems, and its quantitative relation with algebraic connectivity can greatly facilitate the local connectivity awareness (or to a lesser extent, the lower and upper bounds of α), as revealed in the following theorem.

Theorem 1. *Consider graph Laplacian defined in (2) and its corresponding algebraic connectivity α defined in (4). Let $\bar{\alpha}$ and $\underline{\alpha}$ be the upper and lower bounds of α in the sense that $\underline{\alpha} \leq \alpha \leq \bar{\alpha}$. Then, under Assumptions 1 and 2, the bounds $\bar{\alpha}$ and $\underline{\alpha}$ can be formulated as*

$$\begin{aligned} \underline{\alpha}(t) &= \frac{\log \delta(t) + \log n}{-2kt} \\ \bar{\alpha}(t) &= \frac{\log \delta(t) - 2\log(n-1) - \log \sum_{m=1}^{n} m^2}{-2kt} \end{aligned} \tag{11}$$

[3] A group of systems is called the leader (follower) group if they collectively send (receive) information from the rest of the network. Analogously, a system is called a leader (follower) if all communication edges initiated at the underlying system are tails (heads).

[4] Digraph $\mathcal{G}$ is said to be at least connected if it has at least one globally reachable node, which, by definition, is the node that can be reached from any other node by traversing a directed path.

[5] Note that n can also be known locally using a consensus observer.

[6] In numerical mathematics, $\epsilon > 0$ is also known as round-off/measuring error caused by the finite precision of computations or the tolerance associated with the numerical calculators.

The terminal estimation error for (11) is upper bounded, and the convergence time t_f is finite and bounded by

$$t_f \le \frac{2\log\epsilon - 2\log(n-1) - \log\left(\sum_{m=1}^{n} m^2\right)}{-2k\alpha}. \tag{12}$$

Proof. We begin the proof by noticing the analytic solution to system (8) is

$$x(t) = e^{-kL(t-t_0)}x(t_0)$$

where t_0 is initiate time stamp and $t_0 = 0$ is assumed hereafter for the sake of notation brevity.

By invoking Lemma 2, the above expression can be further rewritten as

$$x(t) = \mathbf{1}_n\gamma_1^T x(0) + \Gamma_s e^{-k\Lambda_s t} W_s^T x(0) \tag{13}$$

Consequently, it follows from (10) that

$$\delta(t) = x^T(0)W_s e^{-k\Lambda_s^T t}\Gamma_s^T M(\beta)\Gamma_s e^{-k\Lambda_s t}W_s^T x(0) \tag{14}$$

where $M(\beta) = \text{diag}[\gamma_{1,i}] \in \Re^{n\times n}$.

By invoking Assumption 1, we obtain $\|M(\beta)\| = \max_\ell\{\gamma_{1,\ell}\}$, hence $\frac{1}{n} \le \|M(\beta)\| \le 1$ with $\|.\|$ being the standard Euclidean norm, and since $1 \le \|\Gamma_s\| \le \sqrt{n-1}$, we have

$$\frac{1}{n} \le \|\Gamma_s^T M(\beta)\Gamma_s\| \le n-1 \tag{15}$$

Again, since $1 \le \|W_s\| \le \sqrt{n-1}$ and the initial condition defined in (7), we obtain

$$e^{-k\alpha t} \le \|e^{-k\Lambda_s t}W_s^T x(0)\| \le e^{-k\alpha t}\sqrt{n-1}\|x(0)\| \tag{16}$$

Combining the above three relations, yields,

$$\frac{e^{-2k\alpha t}}{n} \le \delta(t) \le e^{-2k\alpha t}(n-1)^2\sum_{m=1}^{n} m^2 \tag{17}$$

Hence, according to the first inequality of (17), we obtain

$$e^{-2k\alpha t} \le n\delta(t) \tag{18}$$

Taking logarithmic operation on the both sides of (18), yields,

$$\alpha(t) \ge \frac{\log\delta(t) + \log n}{-2kt} \tag{19}$$

Analogously for the second inequality of (17), we obtain

$$\alpha(t) \le \frac{\log\delta(t) - 2\log(n-1) - \log\sum_{m=1}^{n} m^2}{-2kt} \tag{20}$$

Therefore, the bounds $\bar{\alpha}$ and $\underline{\alpha}$ can be formulated as in (11). Moreover, if $\mathcal{G}$ is connected with only one globally reachable node, matrix $A(\mathcal{G})$ is reducible but lower triangular complete. According to (10), we have $M(\beta) = \mathbf{I}$ and thus $1 \le \|\Gamma_s^T M(\beta)\Gamma_s\| \le n-1$. Analogously, (17) becomes

$$e^{-2k\alpha t} \le \delta(t) \le e^{-2k\alpha t}(n-1)^2\sum_{m=1}^{n} m^2 \tag{21}$$

It is apparent that the second inequality of (21) shares the same form as of (17). Again, performing logarithmic operation on the first inequality of (21), we have

$$\alpha(t) \ge \frac{\log\delta}{-2kt} \tag{22}$$

since the right-hand side of (22) is always larger or equal to $\underline{\alpha}$ in (11). Hence, the lower and upper bounds described in (11) also complied with the connected digraphs. Moreover, when the desired consensus is achieved (i.e., $t = t_f$), we have

$$\delta(t_f) = e^{-2k\alpha t_f}\left[x^T(0)W_s\Gamma_s^T M(\beta)\Gamma_s W_s^T x(0)\right]$$

Let $\Omega = x^T(0)W_s\Gamma_s^T M(\beta)\Gamma_s W_s^T x(0)$, then, the actual algebraic connectivity at t_f can be calculated as

$$\alpha(t_f) = \frac{\log\Omega - \log\delta(t_f)}{2kt_f} \tag{23}$$

Suppose the terminal estimation errors are defined as $\underline{\xi} = \alpha(t_f) - \underline{\alpha}(t_f)$ and $\bar{\xi} = \alpha(t_f) - \bar{\alpha}(t_f)$. Then, we obtain,

$$\begin{aligned}\underline{\xi} &= \frac{-\log n - \log\Omega}{-2kt_f}\\ \bar{\xi} &= \frac{\log\sum_{m=1}^{n} m^2 + 2\log(n-1) - \log\Omega}{-2kt_f}\end{aligned} \tag{24}$$

Since $\Omega \le (n-1)^2\sum_{m=1}^{n} m^2$, hence, both $\underline{\xi}$ and $\bar{\xi}$ are upper bounded. In addition, to determine the convergence time, we assume that $\delta = \epsilon^2$ at $t = t_f$, and it follows from (17) that,

$$\epsilon^2 \le e^{-2k\alpha t}(n-1)^2\sum_{m=1}^{n} m^2 \tag{25}$$

Then, convergence time t_f is bounded by (12), which completes the proof of Theorem 1. ■

Theorem 1 implies that, in order to know the bounds of α locally, each system should possess knowledge of the absolute state discrepancy, its value is, however, deemed global and impossible to retain in multi-agent systems. Hence, the remaining challenge becomes how to estimate $\delta(t)$ distributively. In this regard, we propose the following two observers at system i, for $i = \{1, \ldots, n\}$

$$\dot{x}_i'(t) = k'\sum_{j=1}^{n} a_{ij}\left[x_j'(t) - x_i'(t)\right],\ \ x_i(0) = i \tag{26}$$

$$\begin{aligned}\dot{\theta}_i(t) &= k_1\sum_{j=1}^{n} a_{ij}\left[\theta_j(t) - \theta_i(t)\right]\\ &\quad + k_2\sum_{j=1}^{n} a_{ij}\left[u_j(t) - \theta_j(t)\right],\ \ \theta_i(0) = 0\end{aligned} \tag{27}$$

where $x_i' \in \Re$ and $\theta_i \in \Re$ are observers' state, u_i are the reference inputs to be specified in Corollary 1, and k', k_1, k_2 are positive gains to be specified. It is obvious that (26) is proposed in lieu of estimating the consensus value of observer (7) (Qu et al., 2014), while (27) is introduced to estimate the weighted average of $u_i(t)$ and subsequently $\delta(t)$, its applications in balancing power microgrids can be found in Zhang, Li, Qi, and Xin (2017). Performance of observers (26) and (27) is summarized into the following lemma, and its brief proof can be found in the Appendix.

Lemma 3. *Consider graph adjacency matrix defined in (1) and observers defined in (26) and (27). Then, under Assumptions 1 and 2, we have*

- $\lim_{k't\to\infty} x_i'(t) = \sum_{m=1}^{n}\gamma_{1,m}m$, *and its convergence time is bounded by (12) with k replaced by k';*
- *if $k_2 \ge 1$ and $k_1 \gg k_2$, we have, at system i*

$$\lim_{[(k_1-k_2)t]\to\infty}\|\theta_i(t) - \sum_{j=1}^{n}\gamma_{1,j}u_j(t)\| \le \frac{\omega\sqrt{n}}{k_2} \tag{28}$$

where ω is the upper limit of $\dot{u}(t)$, that is, $\|\dot{u}(t)\| \le \omega$. ■

In essence, Lemma 3 implies that (26) behaves as the inner-loop for (7), and their convergence can be effectively separated if $k' \gg k$. Moreover, (27) works as a dynamic consensus tracking observer in general and enables real-time distributive awareness of $\sum_{j=1}^{n}\gamma_{1,j}u_j(t)$ in particular, its convergence time and tracking error are manageable by properly adjusting k_1 and k_2. Combining all three observers, $\delta(t)$ and subsequently the bounds on α can be estimated locally, as summarized in the following Corollary, whose proof is trivial and thus omitted.

Corollary 1. *Consider the same setup as of Theorem 1, suppose $\underline{\hat{\alpha}}^i$ and $\hat{\bar{\alpha}}^i$ are the estimated lower and upper bounds of α at system i, and let $u_i(t) = [x_i(t) - x_i'(t)]^2$ for observer (27). Then, we obtain*

$$\underline{\hat{\alpha}}^i(t) = \frac{\log\theta_i(t) + \log n}{-2kt}$$
$$\hat{\bar{\alpha}}^i(t) = \frac{\log\theta_i(t) - 2\log(n-1) - \log\sum_{m=1}^{n} m^2}{-2kt} \quad (29)$$ ■

Remark 2. Note that, if $\mathcal{G}$ is connected with only one globally reachable node, all systems in either block E_{11} or block E_{22} of Lemma 1 will converge, autonomously and asynchronously, to $\sum_{j\in E_{11}}\gamma_{1,j}u_j(t)$ or $\sum_{j\in E_{11}}\gamma_{1,j}\left[x_j - \gamma_1^T x(0)\right]$ to be more exact, if $E_{21} \neq 0$, since $\gamma_{1,j} = 0$ if $j \in E_{22}$. Hence, $\theta_i(t) \le \delta(t)$ for all $i \in \{1, \ldots, n\}$ if $\mathcal{G}$ is connected, and (29) still applies in this case.

Remark 3. As revealed in Theorem 1 and Corollary 1, the proposed algorithm estimates the bounds on α by discovering and exploiting the logarithmic relation between instantaneous state discrepancies and algebraic connectivity. As such, the resulted estimation algorithm is scalable, susceptible to an easy implementation and affordable for a less stringent topological condition. Compared with existing schemes on estimating α directly, the proposed state-discrepancy based strategy achieves a tradeoff between estimation accuracy and topological requirement, in the sense that it can operate with a *at least connected* digraph, while the estimation error is bounded. In addition, calculation of (29) is fully decentralized with significantly reduced NCC requirement (NCC = 3) that is pertained to existing schemes (Gusrialdi & Qu, 2017; Li & Qu, 2013). Moreover, if NCC is a priority by default, its requirement can be further diminished by noticing that, let $\tilde{\delta}^i(t) = \sum_{j\in\mathcal{N}_i}(x_j - x_i)^2$, $\delta(t)$ can be approximated by $\tilde{\delta}^i$ at system i since their disparity becomes negligible at $t = t_f$ should a consensus be expected. In this regard, the NCC is reduced to 1, and the bounds on α can be approximated by

$$\underline{\tilde{\alpha}}^i(t) = \frac{\log\tilde{\delta}^i(t) + \log n}{-2kt}$$
$$\tilde{\bar{\alpha}}^i(t) = \frac{\log\tilde{\delta}^i(t) - 2\log(n-1) - \log\sum_{m=1}^{n} m^2}{-2kt} \quad (30)$$

However, it should be pointed out that the discrepancy between δ and $\tilde{\delta}$ will inadvertently affect the transient performance of both $\tilde{\bar{\alpha}}^i$ and $\underline{\tilde{\alpha}}^i$ before arriving at t_f, albeit their implications shall vanish once $t \to t_f$.[7] Performance of the proposed scheme is examined in the following simple example:

Example 1. Consider the following three simple cases, whose corresponding adjacency matrices are defined as follows:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0 & 0.3 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.2 & 0.3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.4 & 0 \\ 0.1 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.1 & 0.1 & 0.2 \\ 0.4 & 0.2 & 0.1 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0 & 0.4 \\ 0.4 & 0 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.1 & 0.3 \\ 0.4 & 0.3 & 0 & 0 & 0.3 \end{bmatrix}$$

Hence, it is obvious that A_1 is reducible and hence its corresponding $\mathcal{G}$ is connected with $\alpha(L_1) = 0.1762$; A_2 is irreducible and symmetric hence its corresponding $\mathcal{G}$ is undirected and connected with $\alpha(L_2) = 0.6376$; A_3 is irreducible and asymmetric hence its corresponding $\mathcal{G}$ is strongly connected with $\alpha(L_3) = 0.874$. In addition, it should be pointed out that A_3 possesses complex eigen-structure. In simulations, we choose $\epsilon = 10^{-10}$ and $k = 10$, $k_1 = 150$ and $k_2 = 30$ for all cases, and Fig. 1 demonstrates the estimation errors of $\hat{\bar{\alpha}}^i$ and $\underline{\hat{\alpha}}^i$ for cases in (29), and it is clear that the proposed estimation scheme can track the desired algebraic connectivity with acceptable error, and it can be easily verified that the all convergence time in Fig. 1 complied with (12).

4. Conclusion

Distributed estimation of the bounds on algebraic connectivity for a digraph is investigate under mild assumptions on topological condition and communication effort. Instead of traditional power method, a novel state-dependent approach is proposed to study the real time implications of the algebraic connectivity on consensus procedure and to establish a quantitative relation between convergence rate and estimation accuracy. It is shown that the proposed scheme can solve distributed estimation problem with acceptable performance, for any connected digraph with mild computational and communication effort.

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Appendix. Proof of Lemma 3

The proof of the first statement can be found in *Theorem 2* of Qu et al. (2014). To prove the second statement, we rewrite (27) into a compact form as follows

$$\dot{\theta}(t) = -L_\theta\theta(t) + k_2 A(\mathcal{G})u(t) \quad (31)$$

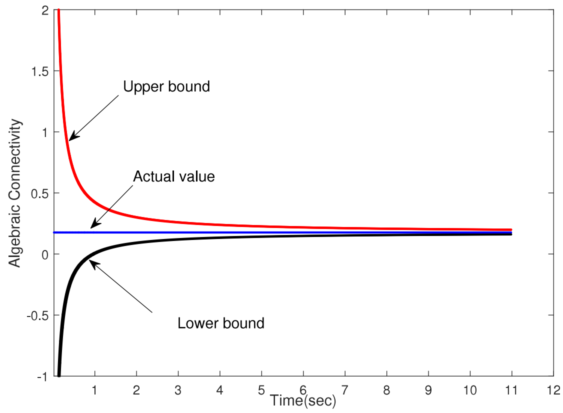
where $\theta = [\theta_1 \ldots \theta_n]^T \in \Re^n$, $u = [u_1 \ldots u_n]^T \in \Re^n$, and $L_\theta = k_2 I + (k_1 - k_2)L$. Then, we have

$$\theta(t) = e^{-L_\theta t}\theta(0) + k_2 e^{-L_\theta t}\int_0^t e^{L_\theta\tau}A(\mathcal{G})u(\tau)d\tau \quad (32)$$

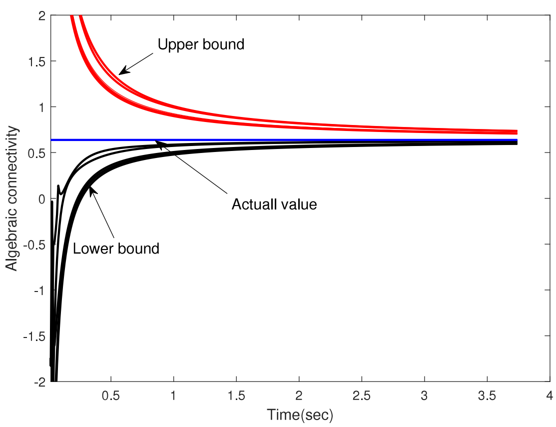
Substituting (6) into (32) and considering the fact that $(\mathbf{1}_n\gamma_1^T)^2 = \mathbf{1}_n\gamma_1^T$, $\Gamma_s e^{(k_1-k_2)\Lambda_s t}W_s^T\mathbf{1}_n\gamma_1^T = 0$ and $\mathbf{1}_n\gamma_1^T A(\mathcal{G}) = \mathbf{1}_n\gamma_1^T$, (32) becomes

$$\begin{aligned}\theta(t) &= e^{-k_2 t}\left[\mathbf{1}_n\gamma_1^T + \Gamma_s e^{-(k_1-k_2)\Lambda_s t}W_s^T\right]\theta(0) \\ &+ k_2 e^{-k_2 t}\int_0^t e^{k_2\tau}\mathbf{1}_n\gamma_1^T u d\tau + k_2 e^{-k_2 t}\mathbf{1}_n\gamma_1^T\int_0^t \\ &e^{k_2\tau}\Gamma_s e^{(k_1-k_2)\Lambda_s\tau}W_s^T A(\mathcal{G})u d\tau + \Gamma_s e^{-(k_1-k_2)\Lambda_s t} \\ &W_s^T e^{-k_2 t}k_2\int_0^t e^{k_2\tau}\Gamma_s e^{(k_1-k_2)\Lambda_s\tau}W_s^T A(\mathcal{G})u d\tau\end{aligned} \quad (33)$$

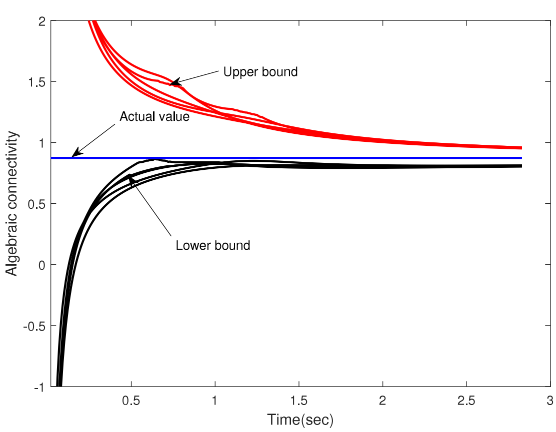
7 We owe this observation to an anonymous reviewer.



(a) A_1



(b) A_2



(c) A_3

Fig. 1. Performance of estimation in Example 1.

Consequently, if we choose $k_2 \geq 1$ and $k_1 \gg k_2$, after several algebraic manipulations, (33) becomes

$$\lim_{[(k_1-k_2)t]\to\infty} \theta(t) = \mathbf{1}_n \gamma_1^T u(t) - e^{-k_2 t} \int_0^t e^{k_2 \tau} \mathbf{1}_n \gamma_1^T \dot{u} d\tau$$

That is,

$$\lim_{[(k_1-k_2)t]\to\infty} \left\| \theta(t) - \mathbf{1}_n \gamma_1^T u(t) \right\| \leq \frac{\omega\sqrt{n}}{k_2} \tag{34}$$

As such, the inequality in (28) can be verified, this concludes the proof. ■

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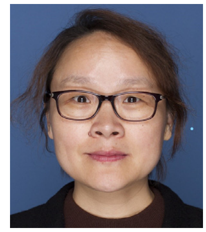

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