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Preserving and Achieving Passivity-Short Property Through Discretization

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***Abstract*—Although passivity has been proven to be quite useful, it excludes a majority of dynamic systems in applications, and typically the property cannot be preserved through discretization. To overcome these shortcomings, the class of passivity-short (PS) systems is investigated in this paper, and canonical forms are derived in both continuous-time and discrete-time domains. Explicit conditions are found to guarantee the PS property through discretizing PS/non-PS systems and, for those that do not satisfy the conditions, two designs of output feedback controls are presented to make sampled-data systems PS regardless of the PS property of original continuous-time systems. One of the designs is in the discrete-time domain, and the other is in the continuous-time domain. The control design to ensure the PS property in the continuous-time domain is also developed. These results provide a full description of PS systems in both domains as well as their relationship through discretization.**

***Index Terms*—Discrete-time Systems, Dissipativity, Linear Systems, Sampled-data Systems, Shortage of Passivity**

I. Introduction

The concept of passivity has become a popular research subject since it was introduced in 1972 [2]. From the viewpoint of the energy dissipation, it allows us to characterize a complex dynamical system in terms of a simple input-output relation, which enables simple analysis and control synthesis. To be specific, a passive system connected through a negative feedback to another strictly passive system (*e.g.,* static gain) is asymptotically stable regardless of their internal dynamics and nonlinearities. Another notable virtue of passivity is that a parallel or a negative feedback interconnection of passive systems are also passive [3], [4]. These properties are powerful tools in analyzing interconnected systems and/or synthesizing controls. Some of the existing results include mechanical systems [5], [6], multi-agent systems [7]–[9], and networked control systems [10], [11].

In such applications as cyber-physical systems, sensors and communication are typically intermittent, and controls must respond. Thus, the control problem in a sampled-data setting is equally important not only in theory but also in practice. While successful applications of passivity have been reported in the continuous-time domain, few results are available in the sampled-data setting since passivity is generally not preserved through discretization [12]–[15].

Despite its popularity, passivity admits only a small subset of dynamic systems. For instance, it calls for minimum-phase and Lyapunov stable systems of relative degree zero or one [16]. In order to relax such constraints, the concept of passivity-short (PS) systems was used to expand the classes of continuous-time systems admissible. Specifically, there are several reasons. First, PS systems include the class of passive systems as a special case and cover much wider classes of systems including nonminimum phase systems. Second, they are simply stablized by an output strictly passive controller (*e.g.,* static output feedback gain). Third, while both negative feedback and parallel connections of two passive systems are also passive, other connections of passive systems do not have such invariance property. In comparison, invariance of PS properties can be established for serial, parallel, negative feedback, and positive feedback connections. These properties make the notion of PS systems useful in stability under interconnection [17]. Furthermore, these properties can be used to modularly synthesize multi-level controls as well as nonlinear controls [18].

In this paper, we apply the concept of PS systems to the discrete-time or discretized systems with a zero-order hold and an ideal sampler. To this end, the class of continuous-time PS systems is first characterized, and an output-feedback control design is presented to achieve the PS property in the continuous-time domain. Then, the PS property in the discrete-time domain is investigated to establish a relationship between the two domains. It is shown that the discretized system is PS when its original continuous-time one is PS and does not have $j\omega$-axis eigenvalues except the origin, but the discretized system becomes non-PS when $j\omega$-axis eigenvalues are present. On the other hand, there is a subclass of non-PS continuous-time systems whose discretized systems are PS. These observations yield two different control designs for continuous-time PS/non-PS system to achieve the PS property for their discretized systems: one in the continuous-time domain and the other in the sampled-data domain. In short, for any stabilizable and detectable continuous-time system, one can always make its discretized system PS, and such property provides much needed tools for analyzing and designing complicated cyber-physical systems.

The rest of this paper is organized as follows. In section II, PS systems are defined in both the continuous and discrete-time domains, and practical examples are included to illustrate the broad applicability of PS systems. In section III, a

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canonical form is revealed for continuous-time PS systems, and a continuous-time control design is presented to achieve the PS property. In section IV, the discrete-time counterparts are provided. In section V, explicit conditions are found under which discretization yields the PS property, and a continuous-time output feedback control design is also provided so that the resulting discretized system is PS. Finally, conclusions are drawn in Section VI.

Notations: For a matrix of complex entities, $A^\star$ and A^T denote the Hermitian and transpose of A, respectively. I_n and $0_{m\times n}$ denote the $n\times n$ identity and $m\times n$ zero matrices, respectively. For a column vector $a=[a_1,\ldots,a_n]^T$, $\text{diag}\{a\}$ or $\text{diag}\{a_i\}$ stand for the diagonal matrix whose ith diagonal entry is a_i. The symbol $\oplus$ denotes the direct sum of matrices. Superscript/subscript shall be omitted if there is no danger of confusion.

II. Concept of Passivity-short and Motivating Examples

In this section, the definition of passivity-short (PS) systems is introduced in both continuous-time and discrete-time domains, and examples are used to show its commonality and wide applicability in typical applications.

Consider the class of linear continuous-time systems:

$$\dot{x}(t)=Ax(t)+Bu(t), \qquad y(t)=Cx(t), \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, and $y\in\mathbb{R}^m$ are the state, input, and output, respectively. Matrices A, B, and C are of appropriate dimensions.

Definition 1: System (1) is said to be dissipative with storage function $V^{\mathrm{c}}(x)$ and supply rate $\Phi^{\mathrm{c}}(x,u)$ if $V^{\mathrm{c}}(x)$ is positive semi-definite and

$$\dot{V}^{\mathrm{c}}(x)\leq\Phi^{\mathrm{c}}(x,u). \tag{2}$$

In addition, system (1) is said to be *passivity-short (PS)* (or, more precisely, *input feedforward passive*) if, for some positive semi-definite function $\eta^{\mathrm{c}}(x)$,

$$\Phi^{\mathrm{c}}(x,u)=-\eta^{\mathrm{c}}(x)+u^Ty+\frac{\varepsilon}{2}\|u\|^2, \tag{3}$$

where constant ε is called the impact coefficient.

The class of linear discrete-time systems is represented as

$$x(k+1)=Fx(k)+Gu(k), \qquad y(k)=Hx(k), \tag{4}$$

where F, G, and H are system matrices of appropriate dimensions. In a similar manner, dissipative discrete-time systems are defined as below.

Definition 2: System (4) is said to be dissipative with storage function $V^{\mathrm{d}}(x)$ and supply rate $\Phi^{\mathrm{d}}(x,u)$ if $V^{\mathrm{d}}(x)$ is positive semi-definite and

$$\Delta V^{\mathrm{d}}(x)\triangleq V^{\mathrm{d}}(x(k+1))-V^{\mathrm{d}}(x(k))\leq\Phi^{\mathrm{d}}(x,u). \tag{5}$$

In addition, system (4) is said to be *passivity-short (PS)* if

$$\Phi^{\mathrm{d}}(x,u)=-\eta^{\mathrm{d}}(x)+u^Ty+\frac{\varepsilon}{2}\|u\|^2, \tag{6}$$

where $\eta^{\mathrm{d}}(\cdot)$ is a positive semi-definite function and ε is a constant.

It is well known that system (1) or (4) is said to be (i) *passive* if $\varepsilon=0$; (ii) *strictly passive* if $\varepsilon<0$. In the above definitions, ε can assume any value in the upper bounding function, which is more general. That is, all systems with $\varepsilon\in\mathbb{R}$ are *passivity-short (PS)*; and, in general, passive systems are special cases of PS systems.

By definition, system (1) or (4) is PS from the control input u to the output y if and only if it is passive from u to the fictitious output $\hat{y}\triangleq y+(\varepsilon/2)u$. Since the system output cannot be redefined in most applications [4], passivity with respect to fictitious output is of little use in practice.

It is also well known that, in the continuous-time domain, passive systems must be of both relative degree one and minimum phase and that even the sampled-data version of the pure integrator is not passive. The following examples show that PS systems do include those of relative-degree two or higher, those of nonminimim phase, and discretized systems. These systems are common in applications.

Example 1: Consider a second-order mass-spring-damper mechanical system as

$$\dot{x}_1=x_2,\quad \dot{x}_2=-x_1-x_2+u,\quad y=x_1, \tag{7}$$

where x_1 and x_2 are the position and velocity respectively, u is the force input, and y is the output. The system is not passive from force input u to position output y_1 since it is of relative degree 2. On the other hand, considering Lyapunov function $V=\frac{3}{2}x_1^2+x_1x_2+x_2^2$, we have

$$\dot{V}=-x_1^2-x_2^2+x_1u+2x_2u\leq uy_1-y_1^2+u^2,$$

which shows that system (7) is PS. $\square$

Example 2: Consider the system:

$$\dot{x}_1=x_2(t)+u(t),\quad \dot{x}_2=-2x_1(t)-3x_2(t)-4u(t),\quad y=x_1. \tag{8}$$

The system has a zero at $s=1$ and hence is nonminimum phase. Choosing Lyapunov function $V=\frac{1}{2}x_1^2+\frac{1}{4}x_2^2$, we have

$$\dot{V}=x_1u-\frac{3}{2}x_2^2-2x_2u\leq uy+\frac{2}{3}u^2.$$

Then, system (8) is PS. $\square$

Example 3: Consider the pure integrator:

$$\dot{x}(t)=u(t), \qquad y(t)=x(t).$$

With the zero-order hold, the discretized system is

$$x(k+1)=x(k)+Tu(k), \qquad y(k)=x(k),$$

where T is the sampling period. It is straightforward to verify using the Lyapunov approach (e.g., $V(k)=0.5\|x(k)\|^2$) that the discretized system is not passive but PS. $\square$

The above examples show that PS systems are much more general, and consequently preserving/achieving the PS property and developing corresponding control designs are of importance for applications. For instance, it has been shown in [17], [18] that modular and plug-and-play controls can be designed for networked PS continuous-time nonlinear systems. In order to synthesize modular and plug-and-play sampled-data controls to networked PS systems, we must investigate the PS property with respect to the true outputs of the systems and by using the standard discretization method of zero-order hold.

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III. Passivity-short Continuous-time Systems and Control Design

In this section, the properties of linear PS continuous-time systems are further explored to obtain a canonical form of PS systems and to develop an output-feedback design of achieving the PS property for any stabilizable and detectable system.

Lemma 1: Suppose that equation (1) is a minimal realization. Then, system (1) is passivity-short (PS) if and only if there exists a state transformation $z = Mx$ such that

$$MAM^{-1} = \begin{bmatrix} \mathsf{A}^s & 0 \\ 0 & \mathsf{A}^m \end{bmatrix} \triangleq \mathsf{A}, \ MB = \begin{bmatrix} \mathsf{B}^s \\ \mathsf{B}^m \end{bmatrix} \triangleq \mathsf{B}, \quad CM^{-1} = \begin{bmatrix} \mathsf{C}^s & \mathsf{C}^m \end{bmatrix} \triangleq \mathsf{C}, \tag{9}$$

where A^s is Hurwitz, A^m has simple eigenvalues on the imaginary axis as, for some non-negative integer p and for McMillan degrees k_j with respect to each distinct imaginary-axis eigenvalue (with $\omega_l \neq \omega_m$ for $l \neq m$)[1],

$$\begin{aligned} &\mathsf{A}^m = \oplus_{j=1}^{p} \mathsf{A}_j^m, \quad \mathsf{A}_1^m = \begin{bmatrix} 0_{k_1 \times k_1} \end{bmatrix}, \\ &\mathsf{A}_j^m = \oplus_{k=1}^{k_j} \begin{bmatrix} 0 & -\omega_j \\ \omega_j & 0 \end{bmatrix}, \quad j = 2, \ldots, p, \end{aligned} \tag{10}$$

and $\{\mathsf{A}^s,\ \mathsf{B}^s,\ \mathsf{C}^s\}$ and $\{\mathsf{A}^m,\ \mathsf{B}^m,\ \mathsf{C}^m\}$ are the minimal realizations corresponding to the stable and marginally-stable eigenvalues of A, respectively, and

$$\begin{aligned} &\mathsf{B}^m = \begin{bmatrix} (\mathsf{B}_1^m)^T & (\mathsf{B}_2^m)^T & \cdots & (\mathsf{B}_p^m)^T \end{bmatrix}^T, \\ &\mathsf{C}^m = \begin{bmatrix} (\mathsf{B}_1^m)^T & (\mathsf{B}_2^m)^T & \cdots & (\mathsf{B}_p^m)^T \end{bmatrix}, \\ &\mathsf{B}_1^m = \begin{bmatrix} \mathrm{diag}\{b_1, \ldots, b_{k_1}\} & 0 \end{bmatrix}. \end{aligned} \tag{11}$$

Furthermore, there exist matrices $\mathsf{P} > 0$, L, W, and constant ε^{c} such that

$$\begin{aligned} &\mathsf{A}^T\mathsf{P} + \mathsf{P}\mathsf{A} = -\mathsf{L}^T\mathsf{L}, \qquad \mathsf{L} = \begin{bmatrix} \mathsf{L}^s & 0 \end{bmatrix}, \\ &\mathsf{B}^T\mathsf{P} - \mathsf{C} = -\mathsf{W}^T\mathsf{L}, \qquad \varepsilon^{\mathrm{c}} I = \mathsf{W}^T\mathsf{W}, \\ &\mathsf{P} = \mathrm{diag}\{\mathsf{P}^s,\ I_{k_1},\ I_{2k_2},\ \cdots,\ I_{2k_p}\}. \end{aligned} \tag{12}$$

The proof of lemma 1 is given in the Appendix. Although it looks involved, lemma 1 is another variation of Kalman-Yakubovich-Popov (KYP) lemma for dissipative systems. It provides a canonical form. It also states that all asymptotically stable systems are PS and also that a PS system must be stable in the sense of Lyapunov, but the converse of either of the statements is not true. The marginally stable part of a PS system has to have the property that, under an appropriate state transformation, $(\mathsf{A}^m,\ \mathsf{B}^m,\ \mathsf{C}^m)$ must satisfy $(\mathsf{B}^m)^T = \mathsf{C}^m$. On the other hand, the asymptotically stable part $(\mathsf{A}^s,\ \mathsf{B}^s,\ \mathsf{C}^s)$ of a PS system is not required to have any particular structural property. In short, for a PS system, its subsystem $(\mathsf{A}^m,\ \mathsf{B}^m,\ \mathsf{C}^m)$ must be passive and its constant ε^{c} is determined solely by its subsystem $(\mathsf{A}^s,\ \mathsf{B}^s,\ \mathsf{C}^s)$.

If system (1) is not PS, one can easily make it PS by designing an output feedback stabilizing controller that achieves asymptotical stability since an asymptotically stable system is always PS. Nonetheless, more needs to be done because, in many applications, systems are desired to have their internal models rather than being asymptotically stable. For instance,

[1]For the McMillan degree, see [19]

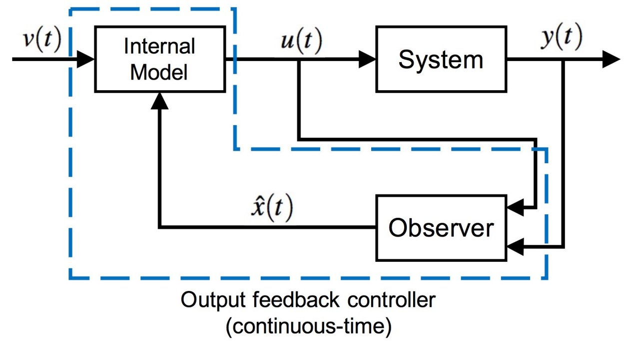

Fig. 1. Continuous-time control design that makes system (1) PS: State observer (13) and dynamic control (14)

the consensus problem of multi-agent systems becomes trivial if each of the systems is made asymptotically stable, and the consensus value is always zero. As such, the consensus problem needs to be solved in a more general setting [7]–[9], [17], [18] and, due to the limitations of passive systems, PS systems are of particular importance [18]. In what follows, we present an analytical design of output feedback control to not only embed an internal model but also achieve the PS property.

As shown in Fig. 1, the output feedback controller is composed of the state observer

$$\begin{aligned} &\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t)), \\ &\hat{y}(t) = C\hat{x}(t), \end{aligned} \tag{13}$$

and the internal model based controller

$$\begin{aligned} &\dot{w}(t) = A_w w(t) + B_w v(t), \\ &u(t) = C_w w(t) + K\hat{x}(t) + v(t), \end{aligned} \tag{14}$$

where $\hat{x}$ is the estimated state of x, $\hat{y}$ is the output of the observer, w is the state of the internal model, L is the observer gain such that $(A - LC)$ is Hurwitz, K is the state feedback gain such that $(A + BK)$ is Hurwitz, and A_w, B_w, and C_w are system matrices which are to be selected according to the following theorem.

Theorem 1: Consider system (1), where the triplet (A, B, C) is assumed to be stabilizable and detectable. Choose C_w such that $BC_w \neq 0$. If all the eigenvalues of A_w are simple and on the imaginary axis[2], there is a unique solution Π to Sylvester equation $\Pi A_w = (A + BK)\Pi - BC_w$. And, under the output feedback controller of (13) and (14) and with $B_w = -(C\Pi)^T$, the continuous-time closed-loop system from the input v to the output y is PS.

Proof: The closed loop system consisting of (1), (13), and (14) is represented as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & BK & BC_w \\ LC & A + BK - LC & BC_w \\ 0 & 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ w \end{bmatrix} + \begin{bmatrix} B \\ B \\ B_w \end{bmatrix} v,$$
$$y = \begin{bmatrix} C & 0 & 0 \end{bmatrix} \begin{bmatrix} x^T & \hat{x}^T & w^T \end{bmatrix}^T.$$

Since $A + BK$ and A_w do not share any common eigenvalue, solution Π to Sylvester equation $\Pi A_w = (A + BK)\Pi - BC_w$ exists uniquely for the given choice of C_w.

[2]Matrix A_w may also be chosen to be Hurwitz, and the corresponding result is similar.

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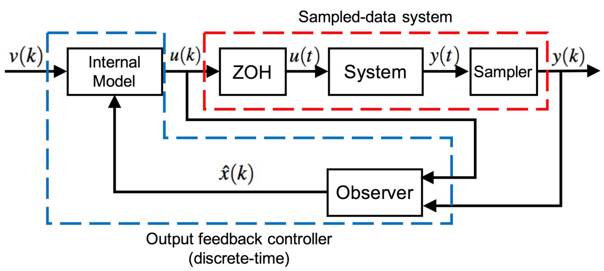

Fig. 2. Discrete-time control design that makes sampled-data system (1) and (15) PS: State observer and dynamic control (21)

Applying the following state transformation

$$\begin{bmatrix} z \\ \tilde{x} \\ w \end{bmatrix} = \begin{bmatrix} I & 0 & \Pi \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ w \end{bmatrix}.$$

we obtain the transformed system as

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{x}} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A+BK & -BK & 0 \\ 0 & A-LC & 0 \\ 0 & 0 & A_w \end{bmatrix} \begin{bmatrix} z \\ \tilde{x} \\ w \end{bmatrix} + \begin{bmatrix} B+\Pi B_w \\ 0 \\ B_w \end{bmatrix} v,$$
$$y = \begin{bmatrix} C & 0 & -C\Pi \end{bmatrix} \begin{bmatrix} z^T & \tilde{x}^T & w^T \end{bmatrix}^T.$$

By lemma 1, we know the above system with input v and output y is PS. ■

IV. Discrete PS Property and Sampled-Data Control Design

In many applications, control is implemented digitally. For networked control systems, hierarchical control (including multi-agent controls) needs to be implemented, and the passivity-short property is critical to ensure not only the stability of the overall system but also the plug-and-play feature of individual systems. Hence, it is more important to analyze and achieve the passivity-short property in the discrete-time domain. To this end, consider the standard sampled-data system configuration in Fig. 2, where the plant has continuous-time dynamics but the output measurement and control action are in the discrete-time domain. Specifically, the control input and the output measurement are given by as

$$u(t) = u(k), \quad y(t) = y(k), \quad kT \le t \le (k+1)T, \tag{15}$$

where k and T are the discrete-time index and sampling period, and $u(k)$ and $y(k)$ denote the input and the output at $t = kT$, respectively. In this case, the discretized version of continuous-time system (1), which is the same form as (4)

$$x(k+1) = Fx(k) + Gu(k), \quad y(k) = Hx(k), \tag{16}$$

where $F = e^{AT}$, $G = (\int_0^T e^{A\tau} d\tau)B$, and $H = C$.

Parallel to the development in Section III, we will subsequently investigate the PS property and the corresponding output feedback control design, both in the discrete-time domain. The following lemma provides necessary and sufficient conditions for system (16) to be PS, and it is the discrete counterpart of lemma 1.

Lemma 2: System (16) is PS if and only if there is an invertible matrix M such that

$$\begin{aligned} MFM^{-1} &= \begin{bmatrix} \mathsf{F}^s & 0 \\ 0 & \mathsf{F}^m \end{bmatrix}, \quad MG = \begin{bmatrix} \mathsf{G}^s \\ \mathsf{G}^m \end{bmatrix}, \\ HM^{-1} &= \begin{bmatrix} \mathsf{H}^s & \mathsf{H}^m \end{bmatrix}, \end{aligned} \tag{17}$$

where F^s has all its eigenvalues inside the unit circle, F^m has only simple eigenvalues on the unit circle as, for non-negative integers p, k_i and for distinct ω_i,

$$\begin{aligned} &\mathsf{F}^m = \oplus_{j=0}^{p} \mathsf{F}_j^m, \quad \mathsf{F}_0^m = -I_{k_0}, \quad \mathsf{F}_1^m = I_{k_1}, \\ &\mathsf{F}_j^m = \oplus_{k=1}^{k_j} \begin{bmatrix} \cos(\omega_j T) & -\sin(\omega_j T) \\ \sin(\omega_j T) & \cos(\omega_j T) \end{bmatrix}, \quad j = 2, \ldots, p, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \mathsf{G}^m &= \begin{bmatrix} (\mathsf{G}_0^m)^T & (\mathsf{G}_1^m)^T & \cdots & (\mathsf{G}_p^m)^T \end{bmatrix}^T, \\ \mathsf{G}_0^m &= \begin{bmatrix} \mathrm{diag}\{g_{01}, \ldots, g_{0k_0}\} & 0 \end{bmatrix}, \\ \mathsf{G}_1^m &= \begin{bmatrix} \mathrm{diag}\{g_{11}, \ldots, g_{1k_1}\} & 0 \end{bmatrix}, \\ \mathsf{H}^m &= \begin{bmatrix} -(\mathsf{G}_0^m)^T & (\mathsf{G}_1^m)^T & \mathsf{H}_2^m & \cdots & \mathsf{H}_p^m \end{bmatrix} \end{aligned} \tag{19}$$

with $(\mathsf{F}_j^m)^T \mathsf{G}_j^m = (\mathsf{H}_j^m)^T$ for $j = 2, \ldots, p$. Furthermore, there exist matrices $\mathsf{P} > 0$, L, W, and constant ε^{d} such that

$$\begin{aligned} &\mathsf{F}^T\mathsf{P}\mathsf{F} - \mathsf{P} = -\mathsf{L}^T\mathsf{L}, \qquad \mathsf{L} = \begin{bmatrix} \mathsf{L}^s & 0 \end{bmatrix}, \\ &\mathsf{F}^T\mathsf{P}\mathsf{G} - \mathsf{H}^T = -\mathsf{L}^T\mathsf{W}, \\ &\varepsilon^{\mathrm{d}} I - \mathsf{G}^T\mathsf{P}\mathsf{G} = \mathsf{W}^T\mathsf{W}, \\ &\mathsf{P} = \mathrm{diag}\{\mathsf{P}^s, I_{k_0}, I_{k_1}, I_{2k_2}, \cdots, I_{2k_p}\}. \end{aligned} \tag{20}$$

The proof of lemma 2 is outlined in the Appendix, and it is a natural extension of the discrete-time KYP lemma in [20] and applicable to general discrete-time linear systems. As can be seen in the third line of (20), system (16) can never be passive unless the corresponding continuous-time system has a direct input feedforward term in the output equation [20]. In other words, in most applications that the relative degree of the continuous-time plant is larger than zero, the discretized system is never passive whether the continuous-time system is passive or not. For those continuous-time systems whose relative degrees are greater than two, their discretized systems always become nonminimum phase for all sufficiently small sampling periods [21] (while passivity requires the minimum phase property). Hence, the concept of PS systems is much more useful for discretized systems as it admits a lot more systems when compared with the concept of passivity. The question whether the PS property can be either preserved or achieved through discretization will be investigated in Section V.

In the rest of this section, a discrete-time output feedback control design is presented to achieve the PS property for discretized systems. Such a result is summarized by the following theorem. Its proof is essentially identical to that of theorem 1 and hence omitted here.

Theorem 2: Consider plant (16) under the dynamic internal-model-based output feedback control

$$\begin{aligned} \hat{x}(k+1) &= F\hat{x}(k) + Gu(k) + L(y(k) - \hat{y}(k)), \\ \hat{y}(k) &= H\hat{x}(k) \\ w(k+1) &= F_w w(k) + G_w v(k), \\ u(k) &= H_w w(k) + K\hat{z}(k) + v(k), \end{aligned} \tag{21}$$

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where L and K are the observer and state feedback gains such that $(F-LH)$ and $(F+GK)$ are Schur stable, respectively. If design matrices F_w, G_w, and H_w are sequentially chosen such that all the eigenvalues of F_w are simple and on the unit circle, that $GH_w \neq 0$, and that $G_w = -(F_w)^{-1}(H\Pi)^T$, where Π is the solution to Sylvester equation $\Pi F_w = (F+GK)\Pi - GH_w$. Then, the overall discrete-time system from input v to output y is PS.

V. Preserving/Achieving PS Property through Discretization

It is well-known that, even if a continuous-time system is passive, its discretized system may not be passive. In fact, the concept of passivity excludes dynamics that involve a time delay [22]. Thus, the conventional discretization using the zero-order holder does not preserve passivity, nor would be any other discretization involving time delays. Accordingly, research has recently been performed to quantify the amount of degradation due to discretization if passivity is lost; see [12]–[15]. Some of these results are derived using the discretization methods other than standard sampling and zero-order hold, *e.g.,* the average or first-order methods [23]–[25]). Another vein of research focuses on the geometric structure-preserving discretization based on the variational integration theory. It can preserve the structural properties such as sympletic, momentum, and long term energy stability, but not the passivity property, and furthermore their discretized systems may become unstable for a large sampling period [26], [27]. To the best of our knowledge, there has not been any study reported on the impact of discretization on the passivity-short property or on preserving relevant properties under the standard discretization method of zero-order hold.

In the sequel, we will explore the avenues of achieving/preserving the PS property for the sampled-data systems. In particular, we will study the effects of sampling on the PS property as well as the control design. We begin with the following theorem on discretizing a continuous-time PS system.

Theorem 3: Consider continuous-time system (1). If system (1) is PS and does not have any pure imaginary eigenvalues (while matrix A may have an eigenvalue at the origin), then the corresponding discretized system (4) is PS. On the other hand, if system (1) is PS and has imaginary eigenvalues $\pm \mathrm{j}\omega$ for $\omega \neq 0$, then the corresponding discretized system (4) is not PS.

Proof: It follows from lemma 1 that system (1) can be transformed into (9) and (10). Hence, under discretization, the matrices corresponding to each of the subsystems can be expressed as

$$\begin{aligned}
&\mathsf{F}^s = e^{\mathsf{A}^s T},\ \ \mathsf{G}^s = (\mathsf{A}^s)^{-1}(\mathsf{F}^s - I)\mathsf{B}^s,\ \ \mathsf{H}^s = \mathsf{C}^s,\\
&\mathsf{F}_1^m = I,\ \ \mathsf{G}_1^m = T\mathsf{B}_1^m,\ \ \mathsf{H}_1^m = \mathsf{C}_1^m,\\
&\mathsf{F}_j^m = \oplus_{k=1}^{k_j} \begin{bmatrix} \cos(\omega_j T) & -\sin(\omega_j T)\\ \sin(\omega_j T) & \cos(\omega_j T)\end{bmatrix},\\
&\mathsf{G}_j^m = \frac{1}{\omega_j}\left(\oplus_{k=1}^{k_j}\begin{bmatrix} \sin(\omega_j T) & \cos(\omega_j T)-1\\ 1-\cos(\omega_j T) & \sin(\omega_j T)\end{bmatrix}\right)\mathsf{B}_j^m,\\
&\mathsf{H}_j^m = \mathsf{C}_j^m,\ \ j = 2,\ldots,p.
\end{aligned} \tag{22}$$

It is well known that stability is preserved under discretization [21]. Thus, the eigenvalues of F^s are inside the unit circle. It follows from lemma 2 that the system composed of $(\mathsf{F}^s, \mathsf{G}^s, \mathsf{H}^s)$ and $(\mathsf{F}_1^m, \mathsf{G}_1^m, \mathsf{H}_1^m)$ is PS. That is, the first statement of the theorem is proven.

On the other hand, the residue of matrix K_j at $e^{\mathrm{j}\omega_j T}$ is

$$\begin{aligned}
e^{-\mathrm{j}\omega_j T}K_j =& \frac{1}{2\omega_j}(\mathsf{G}_j^m)^T \begin{bmatrix} \sin(\omega_j T) & 1-\cos(\omega_j T)\\ \cos(\omega_j T)-1 & \sin(\omega_j T)\end{bmatrix}\mathsf{G}_j^m\\
&+ \frac{\mathrm{j}}{2\omega_j}(\mathsf{G}_j^m)^T \begin{bmatrix} \cos(\omega_j T)-1 & \sin(\omega_j T)\\ -\sin(\omega_j T) & \cos(\omega_j T)-1\end{bmatrix}\mathsf{G}_j^m.
\end{aligned}$$

It is obvious that $e^{-\mathrm{j}\omega_j T}K_j$ is not a positive semi-definite Hermitian. Hence, it follows from lemma 4 in the Appendix that, none of the subsystems $(\mathsf{F}_j^m, \mathsf{G}_j^m, \mathsf{H}_j^m)$, $j = 2,\ldots,p$, has the PS property. Accordingly, the second statement of Theorem 3 is established. ■

The following example is presented to further clarify the implications of Theorem 3.

Example 4: Consider two individual systems:

$$\begin{aligned}
\dot{x}_1(t) &= \begin{bmatrix} 0 & -\omega\\ \omega & 0\end{bmatrix} x_1(t) + \begin{bmatrix} 1\\ 0\end{bmatrix} u_1(t),\\
y_1(t) &= \begin{bmatrix} 1 & 0\end{bmatrix} x_1(t),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\dot{x}_2(t) &= \begin{bmatrix} 0 & -\omega\\ \omega & 0\end{bmatrix} x_2(t) + \begin{bmatrix} \sin(\omega T)\\ 1-\cos(\omega T)\end{bmatrix} u_2(t),\\
y_2(t) &= \begin{bmatrix} 1 & 0\end{bmatrix} x_2(t)
\end{aligned} \tag{24}$$

where ω is a positive constant. It follows from lemma 1 that system (23) is passive (*i.e.,* PS with $\varepsilon_1^{\mathrm{c}} = 0$) and that system (24) is not PS. Their discretized versions are, for $i = 1, 2$,

$$x_i(k+1) = \mathsf{F}_i x_i(k) + \mathsf{G}_i u_i(k), \quad y_i(k) = \mathsf{H}_i x_i(k),$$

where

$$\begin{aligned}
&\mathsf{F}_i = \begin{bmatrix} \cos(\omega T) & -\sin(\omega T)\\ \sin(\omega T) & \cos(\omega T)\end{bmatrix},\ \ \mathsf{H}_i = \begin{bmatrix} 1 & 0\end{bmatrix},\\
&\mathsf{G}_1 = \frac{1}{\omega}\begin{bmatrix} \sin(\omega T)\\ 1-\cos(\omega T)\end{bmatrix},\ \ \mathsf{G}_2 = \frac{2(1-\cos(\omega T))}{\omega}\begin{bmatrix} \cos(\omega T)\\ \sin(\omega T)\end{bmatrix}.
\end{aligned}$$

Applying lemma 2, one can verify through direct computation that the discretized version of system (23) is not PS while the sampled-data version of system (24) is PS with $\varepsilon_2^{\mathrm{d}} = 2(1-\cos(\omega T))/2$. □

Example 4 confirms that, in general, PS is not preserved through discretization for systems that have pure imaginary eigenvalues (other than the origin). In addition, example 4 also shows that certain non-PS continuous-time systems yield discrete PS systems through discretization. The following theorem provides a precise characterization of the PS property from discretizing a non-PS system.

Theorem 4: Suppose that continuous-time system (1) is stable in the sense of Lyapunov and has pure imaginary eigenvalues $\pm \mathrm{j}\omega$ for $\omega \neq 0$. Then, system (1) can be transformed into

$$\begin{aligned}
\dot{\mathsf{x}} &= \mathsf{A}\mathsf{x} + \mathsf{B}u = \begin{bmatrix} \mathsf{A}^s & 0\\ 0 & \mathsf{A}^m\end{bmatrix}\mathsf{x} + \begin{bmatrix} \mathsf{B}^s\\ \mathsf{B}^m\end{bmatrix} u,\\
y &= \mathsf{C}\mathsf{x} = \begin{bmatrix} \mathsf{C}^s & \mathsf{C}^m\end{bmatrix}\mathsf{x},
\end{aligned} \tag{25}$$

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where A^s, B^s, and C^s are the sub-matrices corresponding to the Hurwitz eigenvalues of A, A^m, B^m, and C^m are the sub-matrices related to the simple pure imaginary eigenvalues; specifically

$$
\begin{aligned}
&\mathsf{A}^m = \oplus_{j=1}^{p}\mathsf{A}_j^m, \quad \mathsf{A}_1^m = \left[0_{k_1}\right],\\
&\mathsf{A}_j^m = \oplus_{k=1}^{k_j}\begin{bmatrix} 0 & -\omega_j \\ \omega_j & 0 \end{bmatrix}, \quad j = 2,\dots,p\\
&\mathsf{B}^m = \left[(\mathsf{B}_1^m)^T \quad (\mathsf{B}_2^m)^T \quad \cdots \quad (\mathsf{B}_p^m)^T\right]^T,\\
&\quad \mathsf{B}_1^m = \left[\operatorname{diag}\{b_1,\dots,b_{k_1}\} \quad 0\right],\\
&\mathsf{C}^m = \left[\mathsf{C}_1^m \quad \mathsf{C}_2^m \quad \cdots \quad \mathsf{C}_p^m\right].
\end{aligned}
$$

Then, its corresponding discrete system is PS if and only if

$$
\begin{aligned}
&\mathsf{C}_1^m = (\mathsf{B}_1^m)^T, \quad \mathsf{C}_j^m = (\mathsf{B}_j^m)^T\{(\mathsf{F}_j^m)^T - I\}\{(\mathsf{A}_j^m)^{-1}\}^T\mathsf{F}_j^m,\\
&\mathsf{F}_j^m = \oplus_{k=1}^{k_j}\begin{bmatrix} \cos(\omega_j T) & -\sin(\omega_j T) \\ \sin(\omega_j T) & \cos(\omega_j T) \end{bmatrix}, \; j = 2,\dots,p.
\end{aligned} \tag{26}
$$

Proof: It is well known that block diagonalization (25) can be done. Excluding the subsystems of $(\mathsf{A}_j^m, \mathsf{B}_j^m, \mathsf{C}_j^m)$ for $j = 2,\dots,p$, we know from Theorem 3 that the corresponding discretized system is PS. On the other hand, since A_j^m is invertible for all $j = 2,\dots,p$, G_j^m can be obtained as $\mathsf{G}_j^m = (\mathsf{A}_j^m)^{-1}(\mathsf{F}_j^m - I)\mathsf{B}_j^m$ through discretization [28]. Thus, even if the continuous-time system has simple pure imaginary eigenvalues and is not PS, it is straightforward to verify that the discretized system satisfies lemma 2. ■

For single-input single-output linear systems, one could use the result in [12] to claim that the PS property of asymptotically stable systems (or specifically the positive realness as stated in [12]) is always preserved through discretization. The above theorem shows that the PS property is not preserved for the case that pure imaginary eigenvalues are present. If system (1) has a finite L_2-gain $\gamma > 0$ from input u to the derivative of the output $\dot{y}$, the result of [14] provides a condition on preserving the PS property, which is the same as the first part of Theorem 3. In addition, the requirement of the finite L_2-gain condition in [14] excludes such oscillating systems as a harmonic oscillator studied in the above theorem. In fact, Theorems 3 and 4 can be viewed as the extensions of the previous results to both multi-input multi-output systems and to Lyapunov stable systems.

As can be seen in Theorem 4, continuous-time system (1) needs to have the special structural condition prescribed by (26) in order to make its discretized system PS. Inspired by Theorem 1, we provide another observer-based output feedback controller shown in Fig. 3 in the form of (13) and (14) to satisfy condition (26).

Theorem 5: Consider continuous-time plant (1) under observer-based output feedback controller in the form of (13) and (14). If all the eigenvalues of A_w are chosen to be simple pure imaginary, design matrix C_w is selected such that $BC_w \neq 0$, and if $B_w = -D(C\Pi)^T$ with Π being to the solution to Sylvester equation $\Pi A_w = (A+BK)\Pi - BC_w$, where $\overline{\mathsf{A}}^m = \oplus_{j=2}^{p}\mathsf{A}_j^m$, $\overline{\mathsf{F}}^m = \oplus_{j=2}^{p}\mathsf{F}_j^m$,

$$
D = \begin{bmatrix} I_{k_1} & 0 \\ 0 & \overline{D} \end{bmatrix}, \quad \overline{D} = (\overline{\mathsf{F}}^m - I)^{-1}\overline{\mathsf{A}}^m\overline{\mathsf{F}}^m,
$$

then the discretized system from input v to output y is PS.

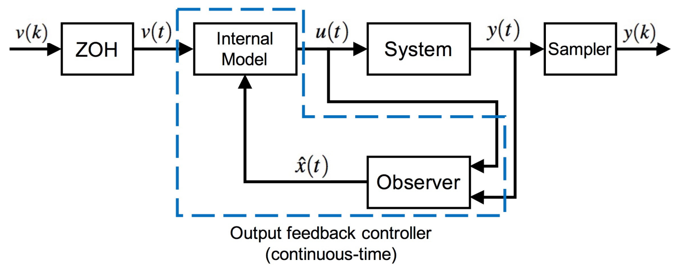

Fig. 3. Continuous-time control design such that closed-loop system (1), (13), and (14) satisfies conditions in Theorem 5

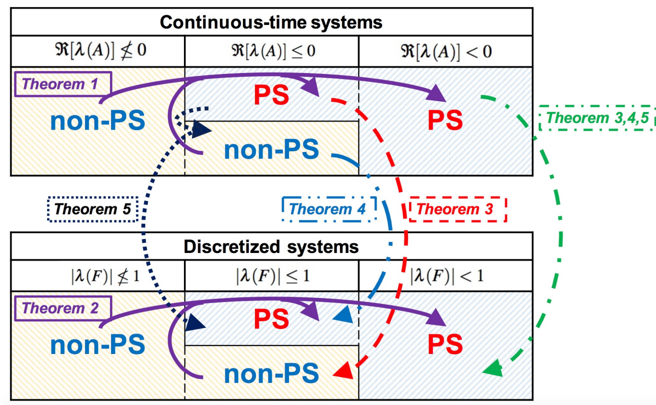

Fig. 4. Preserving/achieving PS under discretization

Theorem 5 is complementary to Theorem 2 to achieve the same outcome, as the former design is in the continuous domain while the latter is in the discrete-time domain. The proof of Theorem 5 is a direct application of those of Theorems 1 and 4, hence it is omitted here.

In this and preceding sections, the means of making continuous-time and sampled-data systems PS are systematically derived and presented: Theorem 1 on the continuous-time control design to make a continuous-time system PS; Theorem 2 on the discrete-time control design to make a discrete-time system PS; Theorem 3 on preserving the PS property from a continuous-time PS system to its discrete-time version; Theorem 4 on the special structure of continuous-time non-PS systems to make its discretized system PS; and Theorem 5 on the continuous-time control design to make a discrete-time system PS. All the results from Theorem 1 to Theorem 5 are illustrated by the Venn diagram in Fig. 4 and on the PS properties.

In order to validate the proposed results, the following examples are presented. The first example is given to illustrate the details of the design procedure in Theorem 5.

Example 5: Consider the following continuous-time system:

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),\\
&y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t).
\end{aligned} \tag{27}
$$

As shown in example 4, system (27) is passive, but its discretized system is not PS.

According to Theorem 5, one can design an output feedback controller so that the discretized closed-loop system

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becomes PS. In particular, matrices K and L are selected as $K = \begin{bmatrix} -3 & -1 \end{bmatrix}$ and $L = \begin{bmatrix} 30 & -199 \end{bmatrix}^T$ so that $(A+BK)$ and $(A-LC)$ are Hurwitz, respectively. Selecting $A_w = A$ and $C_w = C$, we have

$$B_w = -D(C\Pi)^T = \frac{1}{20(1-\cos T)} \begin{bmatrix} 3\sin T - \cos T + 1 \\ -\sin T - 3\cos T + 3 \end{bmatrix}$$

where

$$\Pi = \begin{bmatrix} -0.3 & 0.1 \\ -0.1 & -0.3 \end{bmatrix}, \; D = \frac{1}{2(1-\cos T)} \begin{bmatrix} \sin T & \cos T - 1 \\ 1-\cos T & \sin T \end{bmatrix}.$$

It follows from Theorem 4 that the resulting discretized closed-loop system is PS. Alternatively, one can first discretize system (27) and then apply Theorem 2 to design a discrete-time control to make the resulting system PS. □

The following example shows that, if the PS property is achieved for a plant, its control problems including its network control problem can be readily solved. The sampled-data counterpart is analogous.

Example 6: Suppose that the plant $\mathscr{S}_p$ is PS from input u to output y: for some positive definite V_p,

$$\dot{V}_p \leq u^T y + \frac{\varepsilon}{2}\|u\|^2.$$

Then, the plant becomes L_2 stable under a local static output feedback control:

$$u = v - k_p y,$$

where $k_p > 0$ is a control gain, and $v(t)$ is the control signal received through the network. This is because

$$\dot{V}_p \leq v^T y + \frac{\varepsilon'}{2}\|v\|^2 - \frac{\rho}{2}\|y\|^2, \tag{28}$$

where $\rho = k_p(1-\varepsilon k_p/2) > 0$ holds through choosing k_p, and $\varepsilon' = \varepsilon + 2\varepsilon^2 k_p/(2-\varepsilon k_p)$.

Consider its network control problem in the presence of significant communication time delays (which are not passive). Since the plant is controlled through a network with time delays, the network control v is of form

$$v(t) = r(t-\tau_1) - k_c z(t), \tag{29}$$

where $r(t)$ is the reference input, $z(t) = y(t-\tau_1-\tau_2)$ is the delayed feedback signal, τ_i are the time delays between the plant and its network controller. It follows from (28) and 29 that, by letting $V_f = \int_{t-\tau_1-\tau_2}^t \|y(s)\|^2 ds$,

$$\begin{aligned} \dot{V}_p + k_c\dot{V}_f \quad \leq \quad & r^T(t-\tau_1)y + \frac{\varepsilon'(1+2\varepsilon' k_c)}{2}\|r(t-\tau_1)\|^2 \\ & - \frac{k_c(1-\varepsilon' k_c)}{2}\|z\|^2 - \frac{\rho - 4k_c}{2}\|y\|^2, \end{aligned}$$

which is both L_2 stable and PS if k_c is chosen such that $\varepsilon' k_c < 1$ and $\rho > 4k_c$. □

VI. Conclusion

In this paper, the problem of how to preserve or achieve the passivity-short (PS) property through discretization with a zero-order hold and an ideal sampler is studied. Explicit conditions for preserving the PS property of continuous-time systems through discretization are found, including the canonical forms for PS systems in both the continuous-time and discrete-time domains. Based on these conditions, three output feedback control designs are proposed: a continuous-time control to make systems PS in the continuous-time domain; a discrete-time control to achieve the PS property in the discrete-time domain; and a continuous-time control to yield a PS sampled-data system. These results enable potential applications that, in either continuous and discrete time domains, involve PS systems which are far beyond the class of passive systems.

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APPENDIX

The following lemma is needed to prove lemma 1.

Lemma 3: Let $\Gamma(s)$ be the continuous-time transfer function matrix of system (1). Then, $\Gamma(s)$ is PS if and only if the following two conditions are satisfied:

1) $\Gamma(s)$ is stable in the sense of Lyapunov.
2) For any pure imaginary axis pole $\mathrm{j}\omega$ of $\Gamma(s)$, the residue matrix $\lim_{s\to \mathrm{j}\omega}(s-\mathrm{j}\omega)\Gamma(s)$ is a positive semi-definite Hermitian.

Proof: To show necessity, assume that $\Gamma(s)$ is PS. Since $\Gamma(s)$ is PS, there exists a constant ε^{c} such that $\Gamma(s)+(\varepsilon^{\mathrm{c}}/2)I$ is passive (*i.e.,* positive real [29]). Thus, $\Gamma(s)$ is stable in the sense of Lyapunov and the residue matrix $\lim_{s\to \mathrm{j}\omega}(s-\mathrm{j}\omega)\{\Gamma(s)+(\varepsilon^{\mathrm{c}}/2)I\} = \lim_{s\to \mathrm{j}\omega}(s-\mathrm{j}\omega)\Gamma(s)$ is positive semi-definite Hermitian.

Sufficiency is to be shown next. Since $\Gamma(s)$ is stable in the sense of Lyapunov, $\Gamma(s)$ can be decomposed as $\Gamma(s)=\Gamma^s(s)+\Gamma^m(s)$ where $\Gamma^s(s)$ has poles only in the open left half plane and $\Gamma^m(s)$ has simple imaginary axis poles and is therefore of the form

$$\Gamma^m(s) = \frac{1}{s}\alpha_1 + \sum_{i=2}^{p}\left[\frac{s}{s^2+\omega_i^2}\alpha_i + \frac{1}{s^2+\omega_i^2}\beta_i\right],$$

where α_i and β_i are $m\times m$ real matrices, and $\omega_i>0$ ($\omega_i\neq\omega_j$ for $i\neq j$). Direct calculations using condition 2) in the statement of lemma 3 yield $\alpha_i=\alpha_i^T\geq 0$ and $\beta_i=-\beta_i^T$ for all $i=1,\ldots,p$ (for more details, see lemma 1 in [19]). Thus, for all real ω for which $\mathrm{j}\omega$ is not a pole of any element of $\Gamma(s)$, $\Gamma(\mathrm{j}\omega)+[\Gamma(\mathrm{j}\omega)]^\star = \Gamma^s(\mathrm{j}\omega)+[\Gamma^s(\mathrm{j}\omega)]^\star$. All poles of $\Gamma^s(s)$ are in the open left half plane, so $\Gamma^s(\mathrm{j}\omega)+[\Gamma^s(\mathrm{j}\omega)]^\star$ is bounded. Thus, there exists a constant ε^{c} such that $\Gamma(s)+(\varepsilon^{\mathrm{c}}/2)I$ is positive real. Consequently, $\Gamma(s)$ of system (1) is PS. ■

Proof of Lemma 1: It follows that the transfer function matrix of system (1) is represented as $\Gamma(s)=\Gamma^s(s)+\Gamma^m(s)$. By condition 2) in lemma 3, the residue matrix of $\Gamma^m(s)$ is a positive semi-definite Hermitian, and $\Gamma^m(\mathrm{j}\omega)+[\Gamma^m(\mathrm{j}\omega)]^\star=0$. Hence, $\Gamma^m(s)$ is positive real and its realization can be represented as $(\mathsf{A}^m,\mathsf{B}^m,\mathsf{C}^m)$ in (10) and (11), with the properties of $(\mathsf{A}^m)^T+\mathsf{A}^m=0$ and $(\mathsf{B}^m)^T=\mathsf{C}^m$ (for more details, see [19] and [30]).

In addition, there exists a constant ε^{c} such that $\Gamma^s(s)+(\varepsilon^{\mathrm{c}}/2)I$ is positive real; in other words, P^s, L^s, and W exist for $(\mathsf{A}^s,\mathsf{B}^s,\mathsf{C}^s)$, the realization of $\Gamma^s(s)$, so that $(\mathsf{A}^s)^T\mathsf{P}^s+\mathsf{P}^s\mathsf{A}^s=-(\mathsf{L}^s)^T\mathsf{L}^s$, $(\mathsf{B}^s)^T\mathsf{P}^s-\mathsf{C}^s=-\mathsf{W}^T\mathsf{L}^s$, and $\varepsilon^{\mathrm{c}}I=\mathsf{W}^T\mathsf{W}$. As a result, the realization of $\Gamma(s)$ is represented as equation (9), (10) and (11), and it satisfies the KYP lemma in (12). ■

The following lemma on discretized systems[3] is a counterpart of lemma 3, and it is needed to prove lemma 2.

Lemma 4: Let $\Theta(z)$ be the transfer function matrix for the discretized system of $\Gamma(s)$ in lemma 3. Then, $\Theta(z)$ is PS if and only if the following two conditions are satisfied:

1) $\Theta(z)$ is stable in the sense of Lyapunov.
2) If $z_0=e^{\mathrm{j}\theta_0}$ with real θ_0 is a pole of an entry of $\Theta(z)$ and if K_0 is the residue matrix of $\Theta(z)$ at $z=z_0$, then the matrix $e^{-\mathrm{j}\theta_0}K_0$ is a positive semi-definite Hermitian.

Proof: The proof of necessity is the same as that of lemma 3 except that the discrete-time positive real lemma in [31] needs to be invoked instead of the continuous-time one.

The sufficiency part of the proof is analogous. Since $\Theta(z)$ is stable in the sense of Lyapunov, $\Theta(z)$ can also be decomposed as $\Theta(z)=\Theta^s(z)+\Theta^m(z)$ where $\Theta^s(z)$ has poles inside the unit circle and $\Theta^m(z)$ has simple poles on the unit circle of the complex plane and is of the form

$$\begin{aligned}\Theta^m(z) \;=\; & \frac{1}{z+1}\mu_0+\frac{1}{z-1}\mu_1+\sum_{i=2}^{p}\frac{z}{(z-e^{\mathrm{j}\omega_i T})(z-e^{-\mathrm{j}\omega_i T})}\mu_i \\ & +\sum_{i=2}^{p}\frac{1}{(z-e^{\mathrm{j}\omega_i T})(z-e^{-\mathrm{j}\omega_i T})}\nu_i\end{aligned}$$

where μ_i and ν_i are $m\times m$ real matrices. It follows from condition 2) in lemma 4 that $\mu_0=\mu_0^T\leq 0$, that $\mu_1=\mu_1^T\geq 0$, and that $\mu_i+\mu_i^T=-2\cos(\omega_i T)\nu_i$ and $\nu_i=\nu_i^T\leq 0$ for all $i=2,\ldots,p$. So, for all real ω except for $\omega T=\omega_i$,

$$\begin{aligned}& \Theta(e^{\mathrm{j}\omega T})+[\Theta(e^{\mathrm{j}\omega T})]^\star \\ =\; & \Theta^s(e^{\mathrm{j}\omega T})+[\Theta^s(e^{\mathrm{j}\omega T})]^\star+\mu_0-\mu_1+\sum_{i=2}^{p}\nu_i.\end{aligned}$$

Again, $\Theta^s(e^{\mathrm{j}\omega T})+[\Theta^s(e^{\mathrm{j}\omega T})]^\star$ is bounded since the poles of $\Theta^s(z)$ are inside the unit circle. Therefore, there exists a constant ε^{d} such that $\Theta(z)+(\varepsilon^{\mathrm{d}}/2)I$ is positive real. That is, $\Theta(z)$ is PS. ■

Proof of Lemma 2: The proof goes parallel to that of lemma 1 except that the result on minimal realization of discrete-time positive real systems in [31] and [32] is used. The details are omitted here for briefness. ■

[3]It is also applicable to discrete-time systems in general.